

# Some Contradictions in the Multi-Layer Hele-Shaw Flow

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**Abstract.** An important problem concerning the Hele-Shaw displacements is to minimize the Saffman - Taylor instability. To this end, some constant viscosity fluid layers can be introduced in an intermediate region (*I.R.*) between the displacing fluids. However, we prove that very small (positive) values of the growth rates can be obtained only for a very large (unrealistic) *I.R.* On the contrary, when the *I.R.* length is constrained by certain conditions (for instance, geological), then the maximum value of the growth constants can not fall below a certain value, not depending on the number of layers. This maximum value is not so small.

**Key Words:** Hele-Shaw immiscible displacement; Porous media flow; Linear stability.

## 1. INTRODUCTION

We consider two immiscible fluids that are displacing in a Hele-Shaw cell. Thus we have two regions separated by a sharp interface. On this interface, we have a viscosity jump and in each region we have only one fluid. On the contrary, *in the saturation model*, any point of the medium contains both fluids, each with his own pressure. The difference of the pressures gives the capillary pressure, which appears in the flow equations of this last model.

The displacement of a fluid by a less viscous one in a Hele-Shaw cell is unstable - see [1-4]. The fingering phenomenon and the selection problem in Hele-Shaw displacements are studied in a large number of papers - see [5-7] and references therein.

The optimization of displacements in porous media were studied in [8-12]. An intermediate fluid with variable viscosity in a middle layer between the displacing fluids can minimize the Saffman-Taylor instability - see the experimental and numerical results given in [13-18]. In [19-21] are given theoretical results concerning the linear stability of such three-layer Hele-Shaw flow. Some exact formulas of the growth constants were given, for variable and constant intermediate viscosities. Important results concerning the effects of small surface tensions on interfaces are given in [22, 23].

The Hele-Shaw displacement with  $N$  intermediate layers (the multi-layer Hele-Shaw model) was studied in [24-27] and upper bounds of the growth rates were obtained. The intermediate viscosities with positive jumps in the flow direction (see [24]) lead to very small growth rates of perturbations, if  $N$  is large enough.

In this paper, we study the above multi-layer model in a slightly different form. We do not add many intermediate layers, as in [24]. On the contrary, we consider a *given* intermediate region (*I.R.*), which we divide into  $N$  small layers. We get two main new elements:

1) We get a new formula the growth rates, by using dimensionless quantities and a new eigenfunction in the stability system. The corresponding maximum value not depends on  $N$ , but also not depends on the surface tensions. This last result is in contradiction with the Saffman-Taylor criterion. The new maximum value can not fall below some value which depends only on the ratio of the initial viscosities (the mobility  $M$  given by the first relation (\*) in Introduction ). As a consequence, the maximum growth rate can not be arbitrary small for large enough  $N$ , in contradiction with the results given in [24-27]. Then the multi-layer Hele-Shaw model is useless.

2) In [24-27] were used only dimensional quantities and the maximal value of the growth rates is depending on  $N$  and surface tensions. If we use the dimensionless quantities (as in our paper), the maximal growth rates given in in [24-27] can be very small, but only if  $N$  is unrealistically high. More precisely, we introduce the small parameter  $\varepsilon = b/l$ , where  $b, l$  are the cell gap and length. The growth rates become smaller than  $\varepsilon$  only iff  $N \approx \varepsilon^{-4/3}$ , when the surface tensions are of order 1. Therefore, for  $\varepsilon = 10^{-3}$  we need  $N \approx 10^4$ . Even for a large oil field, this strategy seems to be unrealistic.

## 2. THE THREE-LAYER FLOW WITH VARIABLE INTERMEDIATE VISCOSITY

The flow with variable intermediate viscosity was first described in [14, 15]. However, we recall here some details, very useful for the dimensionless form of the main equations (given in section 3).

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The Hele-Shaw cell is parallel with the  $xOy$  plane. An intermediate region between the two initial immiscible fluids is considered, which contains a given amount of polymer-solute. The adsorption, dispersion and diffusion of the solute in the equivalent porous medium are neglected.

During the displacement process, the initial sharp interfaces change over time and the fingering phenomenon appears. We consider small enough time intervals, to avoid large deformations of the initial interfaces.

Mungan [28] used an intermediate polymer-solute with an exponentially-decreasing viscosity (from the front interface) and obtained an almost stable flow. The displacements with variable viscosity in Hele-Shaw cells and porous media are studied in [29, 30]. On page 3 of [31] is considered a linear viscosity profile in a porous medium.

In this paper, the viscosities of the displacing (respectively displaced) fluids are

$$v_w, v_o, \text{ s.t. } M = \frac{v_o}{v_w} > 1, \quad (*)$$

where  $M$  is the mobility ratio. Thus we consider a less viscous displacing fluid.

Suppose the intermediate region is the interval  $x \in (Ut - Q, Ut)$ , which is moving with the constant velocity  $U$  of the displacing fluid far upstream. We have three incompressible fluids with the viscosities  $v_w, v$  (intermediate layer) and  $v_o$ . The flow equations are quite similar to the Darcy's law in a porous medium, with generic viscosity  $v_G$  and permeability  $\eta = b^2/12$ . Then we have  $\nabla p = -\mu_G u$ , where  $p$  is the pressure,  $u = (u, v)$  is the velocity and

$$\begin{aligned} \mu_G = \frac{v_w}{\eta}, x < Ut - Q; \quad \mu_G = \frac{v(x)}{\eta} := \mu(x), \\ x \in (Ut - Q, Ut); \quad \mu_G = \frac{v_o}{\eta}, x > Ut. \end{aligned} \quad (1)$$

The velocities appearing in (1) are the averages (across the cell plates) of the real (effective) fluid velocities - see [1-3].

The intermediate viscosity  $v$  can be considered as a powers series with respect to the concentration  $C$  of the polymer-solute - see [13, 32]. For a dilute solute,  $v$  is linear with respect to  $C$ , therefore it is invertible. We have  $C_t + uC_x + vC_y = 0$  and we get a "continuity" equation for the intermediate viscosity  $v$ :

$$v_t + uv_x + vv_y = 0, \quad (2)$$

where the indices  $t, x, y$  denote the partial derivatives with respect to time and spatial variables.

We study the linear stability of the basic state

$$u = U, v = 0; \quad x = Ut - Q, x = Ut; \quad P_x = -\mu_G U, \quad P_y = 0. \quad (3)$$

On the interfaces we consider the Laplace's law: the pressure jump is given by the surface tension multiplied with the curvature of the interface. Moreover,  $u$  is continuous across the interfaces (which are material). The basic interfaces are straight lines, thus the basic pressure  $P$  is continuous (but his gradient is not). We use the equations (1) - (2), therefore the basic (unknown)  $\mu$  in  $I.R.$  is given by

$$\mu_t + U\mu_x = 0. \quad (4)$$

We introduce the moving reference frame  $\bar{x} = x - Ut, \tau = t$ . The equation (4) leads to  $\mu_\tau = 0$ , then  $\mu = \mu(\bar{x})$ . The middle region becomes the segment  $-Q < \bar{x} < 0$ . In order to avoid the use of the overbar and  $\tau$ , we still use the simpler notation  $x, t$  instead of  $\bar{x}, \tau$ .

The perturbations of the basic velocity, pressure and viscosity are denoted by  $u', v', p', \mu'$  and are governed by the system (see the relations (2.9)-(2.11) from [14], where  $\mu$  is denoted by  $\mu_0$ )

$$p'_x = -\mu\mu' - \mu'U, \quad p'_y = -\mu v', \quad (5)$$

$$u'_x + v'_y = 0, \quad (6)$$

$$\mu'_t + u'\mu_x = 0. \quad (7)$$

The above equations (5)-(7) are linear in disturbances quantities. Then, as is specified in [14], any perturbation can be decomposed into its Fourier components. Thus we can consider the disturbance  $u'$  in the form

$$u'(x, y, t) = f(x)[\cos(ky) + \sin(ky)]e^{\sigma t}, \quad k \geq 0, \quad (8)$$

where  $f(x), \sigma, k$  are the amplitude, the growth constants and the wave numbers.

As we mentioned above,  $u$  is continuous, thus the amplitude  $f(x)$  is continuous. From (5) - (8) we get the Fourier decompositions for the perturbations  $v', p', \mu'$ :

$$\begin{aligned}
 v' &= (1/k)f_x[-\sin(ky) + \cos(ky)]e^{\sigma t}, \\
 p' &= (\mu/k^2)f_x[-\cos(ky) - \sin(ky)]e^{\sigma t}, \\
 \mu' &= (-1/\sigma)\mu_x f[\cos(ky) + \sin(ky)]e^{\sigma t}.
 \end{aligned}
 \tag{9}$$

The cross derivation of the relations (5)<sub>1</sub>, (5)<sub>2</sub> leads us to

$$\mu\mu'_y + \mu'_y U = \mu_x v' + \mu v'_x.
 \tag{10}$$

From (8), (9)<sub>1</sub>, (10) we get the equation which governs the amplitude  $f$  :

$$-(\mu f_x)_x + k^2 \mu f = \frac{1}{\sigma} U k^2 f \mu_x, \quad \forall x \notin \{-Q, 0\}.
 \tag{11}$$

The above equation is given also in [14] - see the relation (2.17), where  $\mu$  is denoted by  $\mu_0$  and  $f$  is denoted by  $\psi$ . Outside the intermediate region we have constant viscosities and (11) becomes

$$-f_{xx} + k^2 f = 0, \quad x \notin (-Q, 0).$$

In the far field we have

$$f(x) = \begin{cases} f(-Q)e^{k(x+Q)}, & \forall x \leq -Q; \\ f(0)e^{-kx}, & \forall x \geq 0. \end{cases}
 \tag{12}$$

Suppose that a viscosity jump exists in a point  $a$ . The perturbed interface near  $a$  is denoted by  $\xi(a, y, t)$ . In the first approximation (like in [14]) we have  $\xi_t = u$ , therefore

$$\xi(a, y, t) = (1/\sigma)f(a)[\cos(ky) + \sin(ky)]e^{\sigma t}.
 \tag{13}$$

We compute the right and left limit values of the pressure in the point  $a$ , denoted by  $p^+(a)$ ,  $p^-(a)$ . To this end, we use  $P$  in the point  $a$ , the Taylor first order expansion of  $P$  near  $a$  and  $p'(a)$  given by (9)<sub>2</sub>. From (3) it follows  $P_x^+(a) = -\mu^+(a)U$ ,  $P_x^-(a) = -\mu^-(a)U$ , so we get

$$\begin{aligned}
 p^+(a) &= P^+(a) + P_x^+(a)\xi + p'^+(a) = \\
 P^+(a) - \mu^+(a) &\left\{ \frac{Uf(a)}{\sigma} + \frac{f_x^+(a)}{k^2} \right\} [\cos(ky) + \sin(ky)]e^{\sigma t},
 \end{aligned}
 \tag{14}$$

$$p^-(a) = P^-(a) - \mu^-(a) \left\{ \frac{Uf(a)}{\sigma} + \frac{f_x^-(a)}{k^2} \right\} [\cos(ky) + \sin(ky)]e^{\sigma t},
 \tag{15}$$

The Laplace's law is  $p^+(a) - p^-(a) = T(a)\eta_{yy}$ , where  $T(a)$  is the surface tension and  $\eta_{yy}$  is the approximate value of the curvature of the perturbed interface. Since  $P^-(a) = P^+(a)$  (see comments after the formula (3)), from the jump relation equations (14) - (15) we get

$$\begin{aligned}
 -\mu^+(a) \left[ \frac{Uf(a)}{\sigma} + \frac{f_x^+(a)}{k^2} \right] + \mu^-(a) \left[ \frac{Uf(a)}{\sigma} + \frac{f_x^-(a)}{k^2} \right] \\
 = -\frac{T(a)}{\sigma} f(a)k^2.
 \end{aligned}
 \tag{16}$$

The growth constant for three-layer case is obtained as follows. We use the notations  $f_1 = f(-Q)$ ,  $f_0 = f(0)$ . From the relations (12) we have

$$f_x^-(0) = kf_1, \quad f_x^+(0) = -kf_0.
 \tag{17}$$

We multiply with  $f$  in the amplitude equation (11), we integrate on  $(-Q, 0)$ , thus from (16), (17) it follows

$$\begin{aligned}
 \sigma &= \frac{S_0 f_0^2 + S_1 f_1^2 + k^2 U \int_{-Q}^0 \mu_x f^2}{\mu_0 k f_0^2 + \mu_W k f_1^2 + I}, \\
 S_0 &= k^2 U [\mu]_0 - k^4 T_0, \quad S_1 = k^2 U [\mu]_1 - k^4 T_1, \\
 \mu_W &= \frac{v_W}{\eta}, \quad \mu_0 = \frac{v_0}{\eta}, \quad \mu(x) = \frac{v(x)}{\eta}, \\
 I &= \int_{-Q}^0 [\mu f_x^2 + k^2 \mu f^2],
 \end{aligned}
 \tag{18}$$

where  $T_0, T_1$  are the surface tensions in  $x=0, x=-Q$  and

$$\begin{aligned}
 [\mu]_0 &= (\mu^+ - \mu^-)_0 = \mu_0 - \mu^-(0), \\
 [\mu]_1 &= (\mu^+ - \mu^-)_1 = \mu^+(-Q) - \mu_W.
 \end{aligned}
 \tag{19}$$

**Remark 1.** From (16) we can recover the Saffman - Taylor formula

$$\sigma_{ST} = \frac{kU(\mu_0 - \mu_W) - T(a)k^3}{\mu_0 + \mu_W}.
 \tag{20}$$

Indeed, we have

$$\begin{aligned}
 \mu^+(a) &= \mu_0; \quad \mu^-(a) = \mu_W; \\
 f(x) &= f(a)e^{k(x-a)}, \quad x \leq a \Rightarrow f_x^-(a) = kf(a); \\
 f(x) &= f(a)e^{-k(x-a)}, \quad x \geq a \Rightarrow f_x^+(a) = -kf(a) \square
 \end{aligned}$$

**Remark 2.** We can consider several intermediate layers with constant concentrations  $C_1, C_2, \dots, C_N$ , moving with the far upstream velocity  $U$ . Thus we obtain a steady flow of  $N$  layers of immiscible fluids with constant viscosities  $\nu_i, i=1,2,\dots,N$ . This is the multi-layer model studied in [24-27].  $W$

**3. THE FOUR-LAYER HELE-SHAW MODEL WITH CONSTANT INTERMEDIATE VISCOSITIES**

We divide  $(-Q,0)$  in two intermediate fluid layers  $(-Q,-Q/2)$  and  $(-Q/2,0)$  with constant viscosities  $\mu_W < \mu_2 < \mu_1 < \mu_O$  such that

$$\mu(x) = \begin{cases} \mu_W, & x < -Q; \\ \mu_2, & x \in (-Q, -Q/2); \\ \mu_1, & x \in (-Q/2, 0); \\ \mu_O, & x > 0. \end{cases} \tag{21}$$

The basic interfaces are  $x_0 = 0, x_1 = -Q/2, x_2 = -Q$ . This time, the amplitude equation is

$$-(\mu f_x)_x + k^2 \mu f = \frac{1}{\sigma} U k^2 f \mu_x, \quad \forall x \notin \{-Q, -Q/2, 0\}. \tag{22}$$

Inside the intermediate region,  $\mu$  is a Heaviside function. The derivative  $\mu_x$  on the interface  $x = x_i$  is a Dirac distribution, thus

$$\int_{-Q}^0 \mu_x f^2 = f^2(x_i)(\mu_1 - \mu_2). \tag{23}$$

The term (23) is not appearing in the formulas of the growth constants given in [24]. We multiply with  $f$  the relation (22), we integrate on  $(-Q,0)$  and we use the notations

$$(FGH)(x) := F(x)G(x)H(x), \quad f_i := f(x_i).$$

From (12) and (21) we have

$$\mu^-(x_2) = \mu_W, \quad \mu^+(x_0) = \mu_O, \quad f_x^-(-Q) = kf_2, \quad f_x^+(0) = -kf_0.$$

The jump relations (16) in the points  $a = x_2, x_1, x_0$  are

$$(\mu^+ f_x^+)(x_2) = \mu_W kf_2^2 + f_2 \frac{Uk^2}{\sigma} (\mu_W - \mu_2) + \frac{T_2}{\sigma} f_2 k^4,$$

$$\begin{aligned} (\mu^+ f_x^+)(x_1) - (\mu^- f_x^-)(x_1) &= f_1 \frac{Uk^2}{\sigma} (\mu_2 - \mu_1) + \frac{T_1}{\sigma} f_1 k^4, \\ -(\mu^- f_x^-)(x_0) &= \mu_O kf_0^2 + f_0 \frac{Uk^2}{\sigma} (\mu_1 - \mu_O) + \frac{T_0}{\sigma} f_0 k^4. \end{aligned} \tag{24}$$

In (24) we use the viscosities  $\nu_i = b^2 \mu_i / 12$  and it follows

$$\begin{aligned} \sigma &= \frac{S_0 f_0^2 + S_1 f_1^2 + S_2 f_2^2 + k^2 U f_1^2 (\nu_2 - \nu_1)}{\nu_O k f_0^2 + I_1 + I_2 + \nu_W k f_1^2}, \\ S_i &= U k^2 [v]_i - k^4 T_i b^2 / 12, \quad i = 0, 1, 2, \\ [v]_2 &= \nu_2 - \nu_W, \quad [v]_1 = \nu_1 - \nu_2, \quad [v]_0 = M - \nu_1, \\ I_1 &= \nu_1 \int_{x_1}^{x_0} (f_x^2 + k^2 f^2), \quad I_2 = \nu_2 \int_{x_2}^{x_1} (f_x^2 + k^2 f^2). \end{aligned} \tag{25}$$

**Remark 3.** In [24] are used only dimensional quantities. We use the following dimensionless quantities denoted with the super indices  $^a$  :

$$\begin{aligned} x^a &= x/Q, y^a = y/Q, f^a = f/U, \varepsilon = b/Q \approx 10^{-3}, \\ u^a &= u/U, \quad v^a = v/U, \quad T^a = T/(v_W U) \\ v^a &= v/v_W, \quad k^a = kQ, \quad \sigma^a = \sigma/(U/Q). \end{aligned} \tag{26}$$

In the rest of this paper we will omit the  $^a$ . Therefore from now on we have  $\nu_O = M$ .

**Remark 4.** The dimensionless intermediate region is the interval  $(-1,0)$ . The relation (25) gives us the dimensionless growth rate, denoted by  $\sigma_2$  (recall  $\mu_W = 1$ ):

$$\begin{aligned} \sigma_2 &= \frac{\sum_{i=0}^{i=2} \{k^2 [v]_i - \varepsilon^2 k^4 T_i / 12\} f_i^2 + k^2 f_1^2 [v]_1}{k M f_0^2 + I_1 + I_2 + k \cdot 1 \cdot f_2^2} \\ I_1 &= \nu_1 \int_{x_1}^{x_0} [f_x^2 + k^2 f^2] dx, \quad I_2 = \nu_2 \int_{x_2}^{x_1} [f_x^2 + k^2 f^2] dx, \\ x_2 &= -1, \quad x_1 = -1/2, \quad x_0 = 0. \end{aligned} \tag{27}$$

The factor  $\varepsilon^2$  in front of the surface tensions  $T_i$  is very important for the stability analysis. In [24] is given a similar formula, but with dimensional quantities, then without the parameter  $\varepsilon$ . The dimensionless Saffman-

Taylor growth rate and its maximal value are obtained from (20) and (26):

$$\sigma_D = \frac{k(M-1) - k^3 T \varepsilon^2 / 12}{M+1} \leq \sigma_{DMax} = \frac{4(M-1)^{3/2}}{3(M+1)\varepsilon\sqrt{T}}. \tag{28}$$

**4. THE  $N$ -LAYER HELE-SHAW MODEL WITH CONSTANT INTERMEDIATE VISCOSITIES**

We divide the intermediate region in  $N$  small layers with equal length  $(1/N)$ . The interfaces are  $x_i = -i/N, i = 0, 1, \dots, N$ . In the layer  $(x_i, x_{i-1})$  we consider the constant viscosity  $\nu(x) = \nu_i$  such that  $M = M, \nu_{N+1} = 1$  (recall  $\nu_W = 1$ ) and

$$\nu_i = M - i(M-1)/(N+1), \quad (\nu^+ - \nu^-)_i = (M-1)/(N+1). \tag{29}$$

The amplitude equations are

$$-(\nu f_x)_x + k^2 \nu f = \frac{1}{\sigma} U k^2 f \nu_x, \quad \forall x \notin \{-i/N\}, \quad i = 0, 1, \dots, N, \tag{30}$$

thus  $-f_{xx} + k^2 f = 0, \forall x \in (x_i, x_{i-1})$ . The growth constants, denoted by  $\sigma_N$ , are obtained just like  $\sigma_2$  - see the formula (27) in section 3:

$$\begin{aligned} \sigma_N &= \frac{\Omega_N + \Theta_N}{kMf_0^2 + \sum_{i=1}^{i=N} I_i + kf_N^2}, \\ \Omega_N &= \sum_{i=0}^{i=N} \{k^2(\nu^+ - \nu^-)_i - k^4 T_i \varepsilon^2 / 12\} f_i^2, \\ \Theta_N &= \sum_{i=1}^{i=N-1} k^2(\nu^+ - \nu^-)_i f_i^2, \\ I_i &= \int_{x_i}^{x_{i-1}} \nu_i (f_x^2 + k^2 f^2), \quad f_i = f(x_i). \end{aligned} \tag{31}$$

An important point of our paper is the following. The flow is unstable if only one solution of (30) is giving us a positive growth constant (in some range of  $k$ ). Inside each layer we consider the particular eigenfunctions  $f(x) = \exp(kx)$ . We prove that, even if the number  $N$  of intermediate layers is very large, the maximum value of the corresponding growth constants (31) is not so small. Moreover, this maximum value is not depending on the surface tensions and  $N$  - see the formula (39) below. On the contrary, the maximum value of the Saffman-Taylor growth rate (20) is depending on  $T$ .

The term 'surface tension' appears in the section 3 of the seminal paper [4], which is entitled: 'The effect of surface tension on stability in the Hele-Shaw cell.'

**4.1. An Upper Bound of  $\sigma_N$**

From (31) we see that  $\sigma_N$  is negative beyond a finite value of  $k$ , then the "dangerous" values of  $k$  are located in a finite (positive) interval.

*Lemma.* If

$$k \in [0, 1], \quad N = 1/(c-a), \quad k^2(c-a)^2 \approx 0, \quad f(x) = e^{kx},$$

then

$$J(a, c) := \int_a^c (f_x^2 + k^2 f^2) \approx (k^2/N) [f^2(a) + f^2(c)]. \tag{32}$$

*Proof.* As  $f(x) = e^{kx}$ , we get  $f_x^2 + k^2 f^2 = (f_x f)_x$  and

$$J(a, c) = (f_x f)(c) - (f_x f)(a) = k[f^2(c) - f^2(a)]. \tag{33}$$

We use the trapezoidal rule for  $F \in C^2(a, c)$ :

$$\begin{aligned} \int_a^c F(x) dx &= \frac{c-a}{2} [F(a) + F(c)] - R, \\ R &= \frac{(c-a)^3}{12} F''(\chi), \quad \chi \in (a, c). \end{aligned}$$

Consider  $F(x) = f^2(x) = e^{2kx}$ , then  $F''(x) = 4k^2 e^{2kx}$ . A small enough  $(c-a)$  gives us an arbitrary small  $R$ . We use (33), thus we have to prove

$$k[f^2(c) - f^2(a)] \approx k^2(c-a)[f^2(a) + f^2(c)]. \tag{34}$$

For this, we neglect  $k^2(c-a)^2$  and use the first order Taylor expansion of  $f(x) = e^{kx}$ :

$$\begin{aligned} f(c) &\approx f(a) + kf(a)(c-a), \quad f^2(c) \approx f^2(a) + 2kf^2(a)(c-a), \\ f^2(c) - f^2(a) &\approx 2kf^2(a)(c-a), \\ f^2(c) + f^2(a) &\approx 2f^2(a) + 2kf^2(a)(c-a). \end{aligned} \tag{35}$$

The approximation (34) is equivalent with

$$k \cdot 2k(c-a) \approx k^2 \cdot (c-a)[2 + 2k(c-a)],$$

which is true because  $k^3(c-a)^2 \leq k^2(c-a)^2 \approx 0 \square$ .

We use *Lemma* for computing the integrals  $I_i$ , with  $(a, c) = (x_i, x_{i-1}), i = 1, 2, \dots, N$ .

We consider  $k \in [0,1]$  and  $N = 1/(c-a) = 10^2$ , then  $(k/N)^2 \leq 10^{-4}$  and the approximation (34) is verified. From (31) we get

$$\sigma_N \approx \frac{G_0 f_0^2 + \sum_{i=1}^{N-1} G_i f_i^2 + G_N f_N^2}{k v_W f_N^2 + k M f_0^2 + \sum_{i=1}^{N-1} (k^2/N) v_i (f_{i-1}^2 + f_i^2)},$$

$$G_i = 2k^2(v^+ - v^-)_i - k^4 T_i \varepsilon^2 / 12, \quad i = 1, 2, \dots, N-1,$$

$$G_j = k^2(v^+ - v^-)_j - k^4 T_j \varepsilon^2 / 12, \quad j = 0, N. \tag{36}$$

An important new element appearing in this formula is based on the Dirac distributions corresponding to the derivative  $v_x$  on the interfaces. It follows that the "middle" terms  $G_i, 1 \leq i \leq N-1$ , are larger, compared with  $G_0, G_N$ .

We recall the well-known inequality

$$B_i, x_i > 0 \Rightarrow \min\left\{\frac{A_i}{B_i}\right\} \leq \frac{\sum_{i=0}^{i=M} A_i x_i}{\sum_{i=0}^{i=M} B_i x_i} \leq \max\left\{\frac{A_i}{B_i}\right\}. \tag{37}$$

From (36) and (37) it follows

$$\sigma_N \leq \frac{2k^2(v^+ - v^-)_{N-1} - k^4 T_{min} \varepsilon^2 / 12}{k^2(v_N + v_{N-1})/N}.$$

As a consequence, in the range  $k \in [0,1]$ , the upper bound of  $\sigma_N$  is a 2-nd order polynomial in  $k$  and not of 3-rd order, as in [4, 24]. We have

$$v_N + v_{N-1} = (2N - 1 + 3M)/(N + 1),$$

then from the last estimate we get

$$\sigma_N < 2(M-1) \frac{N}{2N-1+3M} - k^2 T_{min} \varepsilon^2 \frac{N(N+1)}{12(2N-1+3M)}, \tag{38}$$

$$\sigma_N \leq 2(M-1) \frac{N}{2N-1+3M} \leq \sigma_{NMax} := (M-1). \tag{39}$$

The estimate (38) holds for  $k \in [0,1]$  and large enough  $N$ . The maximum value (39) is not depending on the surface tension  $T_{min}$ . Here is a strong contradiction: it is very natural to have an improvement

of stability when the surface tensions are very large. Moreover, the maximum value (39) can not be arbitrary small for large  $N$ .

We compare the dimensionless growth rates (28) (given by Saffman-Taylor) and (39), when the (dimensionless) surface tension  $T$  is very large. For  $k \in (0,1)$ ,  $M = 100$ ,  $N = 10^2$ ,  $T = 1/\varepsilon^2$ , the relations (28) and (39) give us

$$\sigma_{DMax} = \frac{4(M-1)^{3/2}}{3(M+1)} \approx 13.3, \quad \sigma_{NMax} = (M-1) = 99. \tag{40}$$

Therefore the maximum value of (28) is less than the maximum value of (39) and the Saffman-Taylor displacement is more stable. As a consequence, the multi-layer Hele-Shaw model with constant intermediate viscosities is not useful.

*Remark 5.* We recall the formulas (100) and (110) from [24]. The formulas (100) give the dimensional growth rates. When  $k$  is large enough, the corresponding estimate of the dimensionless growth rates, say,  $\sigma_{PA}$ , is given by the formula below (recall  $v_W = 1$ )

$$\sigma_{PA} \leq \Sigma_{PA} := \max_k \{k(M-1)/(N+1) - k^3 T_i \varepsilon^2 / 12\}. \tag{41}$$

We have

$$\Sigma_{PA} < \varepsilon \Leftrightarrow (N+1) > \varepsilon^{-4/3} \left(\frac{4}{3}\right)^{2/3} \frac{(M-1)^{3/2}}{\sqrt{T_{min}}}, \tag{42}$$

where  $T_{min}$  is the the lower surface tension. Consider

$$T_{min} \approx 1, \quad M-1 \approx 1, \quad \varepsilon \approx 10^{-3}.$$

Then we need at least  $10^4$  intermediate layers in order to obtain growth constants below  $10^{-3}$ . Even for a large oil field, such a large number of intermediate layers seems to be unrealistic. Moreover, the (dimensionless) length of the intermediate region is also  $10^4$ .

### 5. CONCLUSIONS

Some experimental results proved that an intermediate fluid with a variable viscosity between two displacing fluids in a Hele-Shaw cell can minimize the Saffman-Taylor instability - see [13-18].

A continuous function can be approximated by a step function. For this reason, the multi-layer Hele-Shaw model, consisting of  $N$  intermediate fluid layers

with constant viscosities, was studied in [24-27]. An arbitrary small (positive) upper bound of the growth rates were obtained for large enough  $N$ . Only dimensional quantities were used in these papers and the intermediate region was enlarged by adding new fluid layers.

In this paper we study the multi-layer Hele-Shaw displacements when the length of the intermediate region between the two initial fluids is given. The fluid layers are obtained by dividing the intermediate region into smaller sections.

The new formula (36) of the growth rates is obtained in section 4. This formula contains some terms of Dirac type, related to the derivatives of the viscosity across the interfaces in the intermediate region. We use the dimensionless quantities (26). We prove that even if the number of intermediate layers is very large, the maximum value of the growth constants can not fall below a certain value which depends only on the difference of the viscosities of the initial fluids - see (38).

We obtain the upper bound (39) of the growth constant (result which holds only for bounded values of  $k$  and large values of  $N$ ). The maximum value of (39) is not depending on the surface tensions. Then we can not obtain a significant improvement in the stability, even if the surface tensions and  $N$  are very large. From this point of view, the multi-layer method is useless.

In the last part - see Remark 5 - we prove that the model used in [24], but using dimensionless quantities, can give us very small (positive) growth rates. However this stability improvement can be obtained only for a very large (unrealistic) number of intermediate fluids with constant viscosities.

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## NOMENCLATURE

$I.R.$	= intermediate region between the displacing and displaced fluids.
$N$	= number of the intermediate layers.
$b, l$	= cell gap and length; $\varepsilon = b/l \ll 1$ ; $\eta = b^2/12$ .
$\nu_W, \nu_O$	= viscosities of the displacing and displaced fluids.
$M = \nu_O/\nu_W > 1$	= mobility ratio.

$\nu(x)$	= intermediate viscosity.
$\nu_i$	= intermediate viscosities; $\mu = \nu/\eta$ .
$U$	= velocity of the displacing fluid far upstream.
$u, v$	= fluid velocities.
$p$	= fluid pressure.
$C$	= concentration of the intermediate polymer solution.
$Q$	= constant length of ( $I.R.$ )
$P$	= basic pressure.
$u', v', p', \mu'$	= velocity, pressure and viscosity perturbations.
$f$	= amplitude of $u'$ .
$\sigma$	= growth rates.
$k$	= wavenumbers.
$T, T_i$	= surface tensions.
$+, -$	= right and left limit values.
$a$	= superscript for the dimensionless quantities.
$x_i$	= interfaces between the small intermediate layers.

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